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A wave-interaction model for the generation of windrows

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Interactions of suitable pairs of gravity waves in a shear flow are found to give rise to aperiodic or weakly periodic secondary motions. These secondary flows resemble the 'Langmuir vortices' which are associated with the formation of windrows. It seems likely that such wave interactions will play a substantial part in determining the quasi-steady structure of the flow when wind blows over a water surface.

1. Introduction

When sufficiently strong winds blow over bodies of water, there frequently appears on the water surface a fairly regular pattern of streaks, aligned approximately in the direction of the wind. These streaks, commonly called 'windrows', may appear as lines of foam from breaking waves, or as accumulations of other floating matter (leaves, seaweed, ice, etc.). Often, however, they are visible when no floating objects are present, owing to a change in reflectivity of the water surface in the vicinity of the streak; this change is caused by the accumulation near the streak of a film, or 'slick', of surface contaminant (usually oil or natural organic matter) which smoothes the water surface by eliminating short capillary waves.

The first—and still the most comprehensive—experimental investigation of windrows was made by Langmuir (1938), who found them to be associated with an underlying cellular motion in the water. The streaks appear between converging surface currents and above a downward-flowing current, which are produced by alternately rotating longitudinal vortices—nowadays often called 'Langmuir vortices'. This structure is represented diagrammatically in figure 1, which depicts a vertical section of the flow in a plane perpendicular to the wind direction. The converging surface currents are clearly responsible for the accumulation of buoyant foreign matter or for the compression of a film of surface contaminant into the observed longitudinal streaks.

Since Langmuir's experiments, there has been no shortage of theories to explain the occurrence of streaks on water surfaces. In a brief but useful review, Stommel (1951) distinguishes between 'single streaks' of various origins and 'multiple streaks which usually appear in long parallel lines or with a vein-like pattern'. Of these multiple streaks, not all can be classified as 'windrows'. For example, Ewing (1950) and Dietz & Lafond (1950) have reported parallel streaks with orientation bearing no relation to the wind-direction; these are apparently caused by internal waves, and are quite commonly observed running parallel to coastlines.

A notable feature of windrows is their ability to realign themselves quickly in response to a change in wind direction. For example, Langmuir tells of 500 m long bands of seaweed, each 2–6 m wide and 100–200 m apart, which dispersed and then realigned themselves within 20 min following a 90° change in wind direction. These are windrows on a very large scale: more usually, the spacing of the rows is of order 10 m and their realignment takes only a few minutes. Excellent photographs of windrows are reproduced by Stommel (1951) and Ewing (1950).

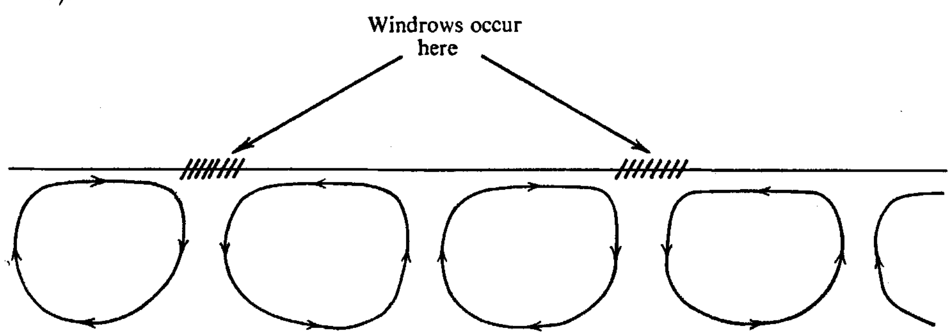

FIGURE 1. Diagramatic representation of Langmuir vortices, showing location of windrows

The generation of windrows is of considerable oceanographic interest, since there appears to be some correlation between the depth of the thermocline and the spacing of the rows (see Langmuir 1938). It is hardly surprising if this is so; for the spacing of the rows is presumably related to the depth of penetration of the Langmuir vortices, and these vortices are likely to provide an efficient mixing mechanism.

While it is recognized that the wind plays a crucial role, there is no real agreement as to the cause of the Langmuir vortices which give rise to windrows.

An attractive suggestion is that they are driven by thermal or thermohaline convection due to surface cooling, and that the orientation of the convection 'rolls' is determined by the shear flow induced by the wind stress. Support to this view is given by Owen (1966) who observed, at sea, rows of horizontal vortices about 1·5 m apart, when no significant wind was present. These were almost certainly thermohaline in origin. Further, it is known (see Gallagher & Mercer 1965; Deardorff 1965) that longitudinal rolls are the preferred configuration of thermal convection in the presence of a shear flow. In short, there seems little doubt that, under suitable conditions, convection-driven rolls do occur. However, experimental investigations by Csanady (1965) and Faller & Woodcock (1964) failed to find evidence of any correlation between the spacing of windrows and the heat flux from the water surface; on the other hand, this spacing is strongly dependent upon wind speed, the rows being wider apart at larger wind speeds. These facts suggest that the cellular structure may have a dynamic, rather than thermal, origin. This view is supported by Stommel (1951), who concluded that, in his cinematographic studies of streaks on ponds, "thermal convection seemed also unlikely because the streaks occurred at times of intense heating of the ponds and with a very stable *epilimnion* (0·1 °C m^{-1})".

Such a dynamic mechanism was proposed by Faller (1964), who suggested that Langmuir vortices are the direct result of hydrodynamic instability of the Ekman boundary layer in the water. He supports this view with experimental observations which appear to show that the orientation of windrows is systematically deflected by a small angle to the right of the wind direction. Similar, but larger, deflexions were previously observed by Faller (1963) in laboratory experiments on the instability of Ekman boundary layers (see also Faller & Kaylor 1967).

Welander (1963) envisages a rather complicated feed-back mechanism whereby a surface film, concentrated near the windrow, reduces the roughness of the water surface by suppressing capillary waves. The associated reduction in drag permits a local increase in wind velocity above the windrow, and this velocity excess drives a secondary spiral motion in the airflow. This motion, in turn, maintains the compression of the surface film. However, on the basis of his cinematographic observations, Stommel (1951) rejects the possibility of a permanent cellular structure in the air, because of its turbulent and gusty nature. Certainly, there is as yet no experimental evidence of such a secondary airflow.

Kraus (1967) has suggested that the surface film is maintained by a radiation stress which derives from the damping of short waves, and that the transverse surface currents result from the generation of waves in the regions between the windrows.

Krauss (1966) has performed a mathematical analysis which suggests that striations in the wind direction might occur when the Coriolis force and pressure gradient balance the Reynolds stresses generated by short-crested gravity waves in a sea with vertically varying current. Ichiye (1967) raised objections to Krauss's model, but also believed that the waves might be important; his heuristic analysis purports to show that streaks may be generated in a manner similar to the Kelvin 'cats-eye' pattern near the critical layer in a shear flow.

Two recent papers, by Stewart & Schmitt (1968) and by Faller (1969), also consider the role of surface waves. The former authors suggest that surface streaks may arise owing to the intersection (without non-linear interaction) of two wave trains. Faller, on the other hand, performs an analysis which has some similarities with that of Krauss (1966) and which suggests that the Reynolds stresses associated with surface waves or turbulence may cause longitudinal rolls. Faller also describes some preliminary laboratory experiments which demonstrate the generation of such rolls by wave action when a shear flow is present.

In another recent paper, McLeish (1968) describes windrows at sea which were irregularly spaced and had frequent intersections. No steady Langmuir circulations were found in the water, and the pattern of streaks changed continuously. McLeish suggests that these streaks may derive from the structure of the turbulence in the water.

Several of the above proposals are based largely on physical arguments, and the details of some of these appear unsatisfactory to the present author. However, it is not the purpose of this paper to speculate concerning the validity of each of the suggested mechanisms. In the final analysis, any hypothesis must remain unproved until supported by firm experimental and theoretical foundations, and it is salutary to recall the words of Stommel (1951) concerning the origin of streaks on water surfaces: "I am personally inclined to admit that under certain circumstances any one of the processes suggested may account for them; to attribute all streaks to the same mechanism would be great a mistake".

When windrows occur, waves are always present and the water undergoes a shearing motion for some depth beneath its surface. Basically, the present paper examines the ingredients of waves and shear flow to determine whether these might induce secondary motions of a type resembling Langmuir vortices.

The physical model to be adopted is similar to that of Craik (1968) in that it concerns gravity-wave interactions in the presence of a uniform shear flow. This model is considerably simpler than those adopted by Krauss (1966) and by Faller (1969). However, although the complexity of the latter models precluded the derivation of detailed solutions such as are found here, the physical basis of these analyses appears to be broadly similar to the present one. The present analysis also has much in common with previous work of Benney (1961, 1964) on the development of streamwise vorticity in unstable boundary layers and shear layers. In this, Benney found that aperiodic secondary motions of longitudinal-vortex type may be induced by the non-linear interaction of suitable pairs of waves, when at least one wave propagates in a direction oblique to the primary flow.

We shall find that interactions of suitable pairs of gravity waves in a shear flow give rise to secondary motions with a structure not unlike Langmuir vortices. In §§3 and 4, two inviscid models are examined to find the initial growth of these secondary motions following the onset of particular wave interactions. In §6, a viscous analysis is presented, which reveals the steady secondary motion driven by the interaction of a suitable pair of waves. Finally the discussion of §8 attempts to relate these results to the observed phenomena: in particular, a plausible criterion is derived for the preferred spacing of windrows.

2. The wave interactions

The flow configuration is shown in figure 2. It is assumed that the wind-induced velocity profile varies linearly with depth z beneath the mean water surface. This approximation is likely to be a reasonable one, since the only relevant portion of the velocity profile is that within a layer near the surface, the depth of which is comparable to the wavelengths of the waves considered. The origin of co-ordinates is chosen in the undisturbed water surface, with the x axis in the direction of the wind and the y axis completing a right-handed Cartesian system. Velocity components in the x, y and z directions are denoted by u, v and w respectively.

The primary velocity profile is then

$$u = -\bar{u}'z, \quad v = w = 0,$$

where $\bar{u}'$ is a positive constant. Also present are gravity waves, each of which has (x, y, t) dependence of the form $\exp[i(kx+ly-\omega t)]$, where $m \equiv (k^2+l^2)^{\frac{1}{2}}$ is the wave-number and ω is the frequency of the wave. In general, there will be no waves of significance propagating upwind, and so we shall consider only those which propagate with a downwind velocity component relative to the liquid surface. Accordingly, we take k and ω to be positive real constants, but l may be either positive or negative. The linear dispersion relation for such gravity waves is (cf. Craik 1968, equation (2.4))

$$\omega = \left\{\left(\frac{k\bar{u}'}{2m}\right)^2 + gm\right\}^{\frac{1}{2}} - \frac{k\bar{u}'}{2m}, \tag{2.1}$$

where g is gravitational acceleration.

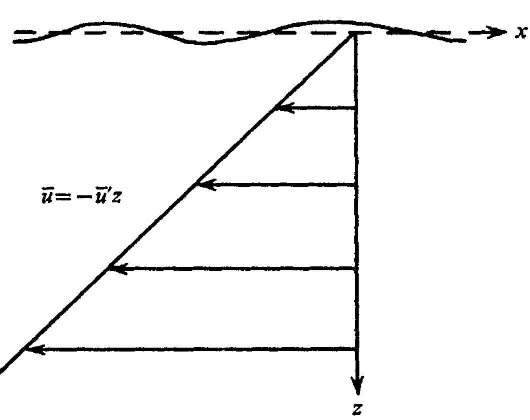

FIGURE 2. Sketch of flow configuration.

Now, consider two waves with wave-number components (k_1, l_1) and (k_2, l_2) respectively, and frequencies ω_1, ω_2 given by the above dispersion relation. The non-linear interaction of such waves yields inertia terms in the equations of motion with (x, y, t) dependence of the form

$$\exp[i\{(k_1x+l_1y-\omega_1t) \pm (k_2x+l_2y-\omega_2t)\}],$$

which may give rise to secondary motions with similar (x, y, t) dependence. However, when $\omega_1 = \omega_2$, one of these secondary components is aperiodic in time; further, if $k_1 = k_2$, this component is also independent of x. The present paper examines the secondary flows which result from such wave interactions. (There are also pairs of gravity waves, for which $\omega_1 = \omega_2$ but $k_1 \neq k_2$, whose interaction may give rise to secondary motions aligned obliquely to the x axis; but it does not seem worthwhile to consider these at present.)

First, we must consider the circumstances under which two waves satisfy the conditions $k_1 = k_2$ and $\omega_1 = \omega_2$. For this, it is convenient to introduce the angle of propagation θ to the direction of the wind, defined as $\theta = \tan^{-1}(l/k)$. Then, from (2.1), two waves with wave-number components (k, l_1) and (k, l_2) have the same frequency ω if

$$\cos\theta_1[(1+K\sec^3\theta_1)^{\frac{1}{2}}-1] = \cos\theta_2[(1+K\sec^3\theta_2)^{\frac{1}{2}}-1] \quad (K = 4gk/\bar{u}'^2).$$

This condition is strictly satisfied, for angles of propagation in the range $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, only when $\theta_1 = -\theta_2$; that is to say, when two waves propagate at equal and opposite angles to the wind direction. However, the condition is *nearly* satisfied by *all* waves, with the same wave-number component k in the x direction, which propagate at *sufficiently small* angles to the wind direction. In the latter case, the relevant non-linear terms of the equation of motion will be only weakly periodic in t, as $\exp[i(\omega_1-\omega_2)t]$, and the resultant secondary motions may also be of interest in the present context.

On the basis of linearized inviscid theory, the velocity components (u_1, v_1, w_1) associated with a surface wave of the form $A_1 \exp[i(kx+ly-\omega t)]$ are given by (cf Craik 1968, equation (2.2)).

$$\left.\begin{aligned} mu_1 &= k\hat{u}_1 - l\hat{v}_1, \quad mv_1 = l\hat{u}_1 + k\hat{v}_1, \quad m^2 = k^2+l^2, \\ \hat{u}_1 &= -iA_1 \exp[-mz]\exp[i(kx+ly-\omega t)], \\ w_1 &= A_1 \exp[-mz]\exp[i(kx+ly-\omega t)], \\ \hat{v}_1 &= \frac{lA_1 e^{-mz}}{imk(z+\alpha)}\exp[i(kx+ly-\omega t)], \quad \alpha = \frac{\omega}{\overline{u}'k}. \end{aligned}\right\} \tag{2.2}$$

The component $\hat{u}_1$ is that in the direction of propagation of the wave, and $\hat{v}_1$ is the 'cross-velocity' component parallel to the wave crests. The components $\hat{u}_1$ and w_1 together comprise an irrotational wave; but the component $\hat{v}_1$ possesses vorticity which derives from the periodic stretching and contraction of the vortex lines in the primary flow. However, contrary to the case examined by Craik (1968), $\hat{v}_1$ here has no singularity at the depth $z = -\alpha$, since $\overline{u}'$ is taken to be positive and the liquid occupies only the region $z \geqslant 0$.

In §§3 and 4, we shall consider, in turn, two classes of wave interaction, which may be called 'symmetric' and 'non-symmetric' respectively. The first involves two similar waves propagating at equal and opposite angles to the wind direction, for which the conditions $k_1 = k_2$ and $\omega_1 = \omega_2$ are exactly satisfied; and the second concerns a two-dimensional wave of wave-number k and a wave propagating at a small angle to the x direction with wave-number components (k, l) where $|l| \ll k$. As noted above, the condition $\omega_1 = \omega_2$ is only approximately satisfied in the latter case. In both these analyses, viscosity is neglected; but, in §6, a viscous analysis is presented for symmetric interactions.

3. The symmetric interaction $l_1 = -l_2$

Consider the interaction of a wave of the form given in (2.2) with another of the form $A_2 \exp[i(kx-ly-\omega t)]$. The velocity components (u_2, v_2, w_2) of the latter wave may be obtained from (2.2) on changing the subscripts and replacing l by $-l$. Without loss of generality, we choose l to be positive. The interaction of these waves produces a secondary motion independent of x, which we represent by the velocity components (u_3, v_3, w_3). By continuity, we have

$$\frac{\partial v_3}{\partial y} + \frac{\partial w_3}{\partial z} = 0,$$

and we may introduce a stream function $\psi \equiv \phi(z,t) \exp[2ily]$ such that

$$v_3 = \frac{\partial \phi_3}{\partial z} \exp[2ily], \quad w_3 = -2il\phi_3 \exp[2ily]. \tag{3.1}$$

The momentum equations for this motion are

$$\left.\begin{aligned} \frac{\partial u_3}{\partial t} - \bar{u}' w_3 &= -[\mathbf{u} . \nabla u]_3, \\ \frac{\partial v_3}{\partial t} + \rho^{-1} \frac{\partial p_3}{\partial y} &= -[\mathbf{u} . \nabla v]_3, \\ \frac{\partial w_3}{\partial t} + \rho^{-1} \frac{\partial p_3}{\partial z} &= -[\mathbf{u} . \nabla w]_3, \end{aligned}\right\} \tag{3.2a–c}$$

where ρ is the density, $p_3(y,z,t)$ is the pressure component proportional to $\exp[2ily]$, and the terms on the right-hand sides are those of the form $\exp[2ily]$ which derive from the interaction of the two waves. The elimination of p_3 from (3.2*b*, *c*) leads to the vorticity equation

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \phi}{\partial z^2} - 4l^2 \phi\right) = 2il[\mathbf{u} . \nabla w]_3 - \frac{\partial}{\partial z}[\mathbf{u} . \nabla v]_3. \tag{3.3}$$

In evaluating the non-linear terms on the right-hand side of this equation, the results of linear theory may be used. However, some care is necessary: in particular, use must be made of the multiplication rule

$$\mathrm{Re}\{A\} \,\mathrm{Re}\{B\} = \tfrac{1}{2}\mathrm{Re}\{AB + AB^*\},$$

where B^* is the complex conjugate of B. After some reduction, it is found that the right-hand side of (3.3) equals

$$ilA_1 A_2^* \exp[2ily - 2mz]\left[\frac{2(3k^2+l^2)}{m^2(z+\alpha)} + \frac{5k^2 - 3l^2}{m^3(z+\alpha)^2} + \frac{2(k^2-l^2)}{m^4(z+\alpha)^3}\right].$$

Since the velocity components v_3 and w_3 must tend to zero at large depths, we have the boundary condition

$$\phi(\infty, t) = 0. \tag{3.4}$$

In addition, there are kinematic and pressure boundary conditions to be satisfied at the water surface.

We denote the position of the water surface by

$$z = \zeta(x,y,t) = \zeta_1 + \zeta_2 + \zeta_3 + \ldots,$$

where the ζ_i are the displacements associated with each periodic disturbance: thus, $\zeta_{1,2}$ are the surface displacements of the waves $A_{1,2} \exp[i(kx \pm ly - \omega t)]$, and ζ_3 is the displacement proportional to $\exp[2ily]$ corresponding to the secondary velocity components v_3 and w_3. Assuming that the pressure component proportional to $\exp[2ily]$ is zero at the surface $z = \zeta$, we have the boundary condition

$$p_3 + \rho g \zeta_3 = -\frac{1}{2}\left(\zeta_1 \frac{\partial p_2^*}{\partial z} + \zeta_2^* \frac{\partial p_1}{\partial z}\right) \quad (z = 0), \tag{3.5}$$

where p_3 is the pressure component associated with v_3 and w_3, and where the non-linear terms derive from the interaction of the two waves. Also, the corresponding kinematic surface condition is

$$\frac{\partial \zeta_3}{\partial t} - w_3 = \frac{1}{2}\left[\zeta_1 \frac{\partial w_2^*}{\partial z} + \zeta_2^* \frac{\partial w_1}{\partial z} - \left(u_2^* \frac{\partial}{\partial x} + v_2^* \frac{\partial}{\partial y}\right)\zeta_1 - \left(u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y}\right)\zeta_2^*\right] \quad (z = 0). \tag{3.6}$$

Now, if the results (2.2) of linear theory are used to evaluate the non-linear terms of these relations, it is found that the right-hand side of (3.6) is identically zero, and that the right-hand side of (3.5) is independent of time t. We therefore obtain the simple result

$$w_3 = -\frac{1}{\rho g}\frac{\partial p_3}{\partial t} \quad (z = 0), \tag{3.7}$$

from which the nonlinear terms have vanished identically. Further, equation (3.2*b*) yields the relationship

$$\frac{\partial^2 v_3}{\partial t^2} + \frac{2il}{\rho}\frac{\partial p_3}{\partial t} = 0,$$

and these two results combine to give the final boundary condition, that

$$\frac{\partial^2 v_3}{\partial t^2} - ilgw_3 = 0 \quad (z = 0). \tag{3.8}$$

It must be emphasized that the non-linear terms resulting from the wave interaction have not been neglected in deriving this boundary condition; these terms vanish identically when they are evaluated by means of linear theory.

If we now suppose $\phi(z,t)$ to be of the form

$$\phi(z,t) = tA_1A_2^* \frac{il}{m^4} f(z), \tag{3.9}$$

the equation (3.3) and boundary conditions (3.4) and (3.8) reduce to

$$\frac{d^2 f}{dz^2} - 4l^2 f = e^{-2mz}\left[\frac{2m^2(3k^2+l^2)}{z+\alpha} + \frac{m(5k^2-3l^2)}{(z+\alpha)^2} + \frac{2(k^2-l^2)}{(z+\alpha)^3}\right] \equiv G(z)$$

$$f(\infty) = 0, \quad f(0) = 0. \tag{3.10}$$

The solution for $f(z)$ is then

$$f(z) = \frac{1}{4l}\left[e^{-2lz}\int_0^\infty G(\zeta)\, e^{-2l\zeta}\, d\zeta - e^{-2lz}\int_0^z G(\zeta)\, e^{2l\zeta}\, d\zeta - e^{-2lz}\int_z^\infty G(\zeta)\, e^{-2l\zeta}\, d\zeta\right], \tag{3.11}$$

and evaluation of the integrals yields

$$f(z) = (\tfrac{1}{2}m^3 + 2lm^2 - 2l^3)\, e^{2(m-l)\alpha}\, e^{-2lz}\,[E_1\{2(m-l)(z+\alpha)\} - E_1\{2(m-l)\alpha\}]$$
$$+ (\tfrac{1}{2}m^3 - 2lm^2 + 2l^3)\, e^{2(m+l)\alpha}\,[e^{2lz} E_1\{2(m+l)(z+\alpha)\} - e^{-2lz} E_1\{2(m+l)\alpha\}], \tag{3.12}$$

where $$E_1\{\mu\} \equiv \int_\mu^\infty \frac{e^{-\xi}\, d\xi}{\xi}.$$

We note that, when μ is large, $\mu e^{\mu} E_1\{\mu\} \sim 1$: therefore, when $2(m-l)\alpha \gg 1$, a good approximation for all z is found to be

$$f(z) = \frac{(m^4+4k^2m^2-4k^4)}{2k^2}\left(\frac{e^{-2mz}}{z+\alpha}-\frac{e^{-2lz}}{\alpha}\right). \tag{3.13}$$

Clearly, $f(z)$ may now be calculated, from either (3.12) or (3.13), for particular values of m, l and α. This has been done for the case $m = 0{\cdot}1\,\mathrm{cm}^{-1}$, $l = 0{\cdot}02\,\mathrm{cm}^{-1}$ and $\alpha = 10^2\,\mathrm{cm}$, which represents waves approximately 60 cm in length propagating at angles of $11\frac{1}{2}°$ to the wind direction, in the presence of a velocity gradient $\bar{u}'$ of about $1\,\mathrm{sec}^{-1}$. The functions $f(z)$ and $f'(z)$ for this case are shown in figure 3.

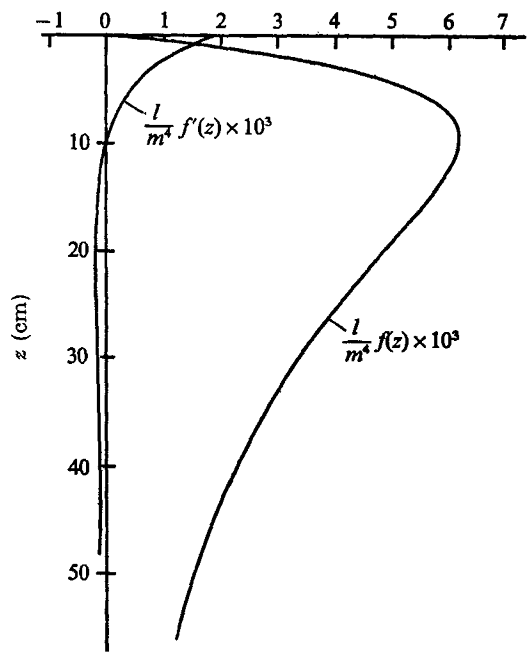

FIGURE 3. The functions $f(z)$ and $f'(z)$ for the case $m = 0{\cdot}1\,\mathrm{cm}^{-1}$, $l = 0{\cdot}02\,\mathrm{cm}^{-1}$, $\alpha = 100$ cm.

The complex stream function for the secondary velocity components v_3, w_3 is

$$\psi(y,z,t) = tA_1A_2^*(il/m^4)f(z)\,e^{2ily}.$$

This motion, which is initially zero, grows linearly with time; also, the streamlines in the y–z plane satisfy equations of the form

$$\mathrm{Re}\{A_1A_2^*(il/m^4)f(z)\,e^{2ily}\} = \text{constant}.$$

Further, the location of the axis $y = 0$ may be chosen, without loss of generality, so that $(iA_1A_2^*)$ is real and the streamlines are given by

$$f(z)\cos 2ly = \text{constant}.$$

For the case defined above, such streamlines are shown in figure 4.

It remains to calculate the velocity component u_3 in the x direction. This is readily obtained from equation (3.2*a*), since the non-linear terms vanish identically when evaluated by linear theory, and we are left with the simple result

$$\partial u_3/\partial t = \bar{u}' w_3.$$

This, together with (3.9), gives

$$u_3 = \mathrm{Re}\left\{t^2\left(\frac{\bar{u}' l^2}{m^4}\right) A_1 A_2^* f(z)\, e^{2ily}\right\}, \tag{3.14}$$

showing that u_3 grows as t^2 instead of t, and that the dependence on z is identical with that of w_3.

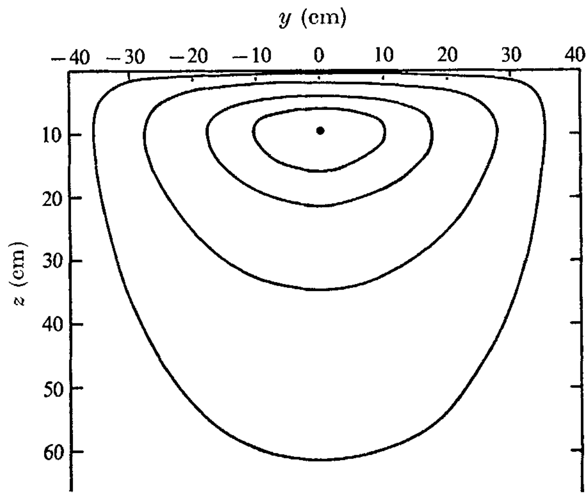

FIGURE 4. The streamlines $f(z)\cos 2ly$ = constant, for the symmetric interaction $m = 0{\cdot}1\ \mathrm{cm}^{-1}$, $l = 0{\cdot}02\ \mathrm{cm}^{-1}$, $\alpha = 100$ cm.

4. A non-symmetric interaction

Suppose, now, that a two-dimensional wave $A_1 \exp[i(kx-\omega t)]$ interacts with an oblique wave $A_2 \exp[i(kx-ly-\omega' t)]$, where $k \gg l$ and, for convenience, l is taken to be positive. Then, as shown in §2, the frequencies ω and ω' are nearly, but not quite, equal, with ω' slightly larger than ω. Accordingly, we define $\Delta\omega = \omega' - \omega$. The velocity components (u_1, v_1, w_1), (u_2, v_2, w_2) associated with the waves are readily obtained from (2.2). (It is seen immediately that $v_1 = 0$.) The interaction of these waves yields some non-linear inertia terms of the form $A_1 A_2^* \exp[i(ly+\Delta\omega t)]$, which vary only slowly with time. We represent the resultant secondary motion by the velocity components (u_3, v_3, w_3), and we define a stream function $\psi \equiv \phi(z,t)\, e^{ily}$ such that

$$v_3 = \frac{\partial\phi}{\partial z} e^{ily}, \quad w_3 = -il\phi\, e^{ily}. \tag{4.1}$$

The equations of motion (3.2*a*–*c*) now yield the result

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\phi}{\partial z^2} - l^2\phi\right) = il[\mathbf{u}\,.\,\nabla w]_3 - \frac{\partial}{\partial z}[\mathbf{u}\,.\,\nabla v]_3, \tag{4.2}$$

and the non-linear right-hand side may be evaluated by linear theory as

$$ilA_1A_2^* e^{i\Delta\omega t} e^{-(m+k)z}\left[\frac{(2m^2+mk-k^2)}{2m^2(z+\alpha)}+\frac{k+2m}{2m^2(z+\alpha)^2}+\frac{1}{m^2(z+\alpha)^3}\right].$$

Also, it may be shown that the boundary conditions at the liquid surface yield, as the result equivalent to (3.8),

$$\frac{\partial^2 v_3}{\partial t^2}-ilgw_3 = \frac{gl^3}{2\omega m}\left(1+\frac{1}{m\alpha}\right)A_1A_2^*\exp\left[i(ly+\Delta\omega t)\right] \quad (z=0). \tag{4.3}$$

We note that, in this case, the non-linear terms are not now identically zero. The boundary condition at infinity is again

$$\phi = 0 \quad (z=\infty). \tag{4.4}$$

Some difficulties are encountered if, as was done in §3, the solution $\phi(z,t)$ is required to vanish at $t=0$. For this to be so, $\phi(z,t)$ must take the form

$$\phi = \frac{lA_1A_2^*}{m^4\Delta\omega}(e^{i\Delta\omega t}-1)f(z)$$

(provided, of course, that linear gravity-wave components with frequency $(gl)^{\frac{1}{2}}$ are assumed to be absent). The function $f(z)$ must then satisfy the equation

$$\frac{\partial^2 f}{\partial z^2}-l^2 f = m^2 e^{-(m+k)z}\left[\frac{2m^2+mk-k^2}{2(z+\alpha)}+\frac{2m+k}{2(z+\alpha)^2}+\frac{1}{(z+\alpha)^3}\right] \equiv G(z), \tag{4.5}$$

while (4.3) and (4.4) yield the *three* boundary conditions

$$f(0) = 0, \quad f(0)+\frac{(\Delta\omega)^2}{l^2 g}f'(0) = -\frac{\Delta\omega m^3}{2\omega}\left(1+\frac{1}{m\alpha}\right), \quad f(\infty)=0. \tag{4.6a–c}$$

Clearly, the boundary conditions are over-specified and no solution (fz) can be found.

However, we may find a solution denoting an *initially irrotational* secondary flow, which subsequently develops vorticity. If we write

$$\phi = \frac{lA_1A_2^*}{m^4}\left\{\frac{(e^{i\Delta\omega t}-1)}{\Delta\omega}f(z)+h(z)\right\},$$

and require that the associated disturbance is irrotational at $t=0$, $h(z)$ must satisfy the equations and boundary conditions

$$\frac{\partial^2 h}{\partial z^2}-l^2 h = 0, \quad h(0) = (\Delta\omega)^{-1}f(0), \quad h(\infty)=0, \tag{4.7a–c}$$

and $f(z)$ must satisfy equation (4.5) and the two boundary conditions (4.6b,c). This now constitutes a well-posed problem.

When $\Delta\omega$ is sufficiently small, the boundary conditions (4.6b,c) and (4.7b,c) may be approximated by

$$f(0) = 0, \quad f(\infty) = 0; \qquad h(0) = -\frac{m^3}{2\omega}\left(1+\frac{1}{m\alpha}\right), \quad h(\infty)=0. \tag{4.8}$$

(Note that, to this approximation, a solution $\phi(z,t)$ which is zero at $t = 0$ may also be found, since (4.6a, b) become identical.)

The functions $f(z)$ and $h(z)$ are both real, and the solutions which satisfy the boundary conditions (4.8) are found to be

$$h(z) = -\frac{m^3}{2\omega}\left(1+\frac{1}{m\alpha}\right)e^{-lz},$$

$$\frac{2}{m^2}f(z) = \frac{e^{-(m+k)z}}{z+\alpha} - \frac{e^{-lz}}{\alpha} - (l-\tfrac{1}{2}k)\,e^{(m+k-l)\alpha}\,e^{-lz}$$
$$\times[E_1\{(m+k-l)\,\alpha\} - E_1\{(m+k-l)\,(z+\alpha)\}] + (l+\tfrac{1}{2}k)\,e^{(m+k+l)\alpha}$$
$$\times[e^{-lz}E_1\{(m+k+l)\,\alpha\} - e^{lz}E_1\{(m+k+l)\,(z+\alpha)\}]. \tag{4.9a,b}$$

When $(m+k-l)\,\alpha \gg 1$, a good approximation for $f(z)$ is

$$f(z) = \frac{m^2(2m-k)}{4k}\left[\frac{e^{-(m+k)z}}{z+\alpha} - \frac{e^{-lz}}{\alpha}\right]. \tag{4.10}$$

The streamlines for the motion in the y–z plane are given at any instant by

$$\mathrm{Re}\left\{A_1A_2^*\left[\left(\frac{e^{i\Delta\omega t}-1}{\Delta\omega}\right)f(z)+h(z)\right]e^{ily}\right\} = \text{constant}.$$

The secondary flow is clearly periodic in time with frequency $\Delta\omega$. Since $\Delta\omega/\omega \ll 1$, the inequality

$$|(e^{i\Delta\omega t}-1)f(z)| \gg \Delta\omega|h(z)|$$

is satisfied for all t not too close to $2\pi n/\Delta\omega$, where n is integer, and for all z not too near zero. If also we choose $A_1A_2^*$ to be real, the approximate streamlines at any instant are given by

$$f(z)\,[\cos{(ly+\Delta\omega t)} - \cos{ly}] = \text{constant}.$$

This flow may conveniently be regarded as that for two identical systems of alternately rotating longitudinal vortices, of which one is stationary and the other travels with velocity $\Delta\omega/l$ in the negative y direction. For the particular case $m = 0{\cdot}1\,\mathrm{cm}^{-1}$, $l = 0{\cdot}02\,\mathrm{cm}^{-1}$, $\alpha = 20{\cdot}2\,\mathrm{cm}$, the lines on which $f(z)\cos{ly}$ is constant are shown in figure 5. For this case, $\overline{u}' = 4{\cdot}04\,\mathrm{sec}^{-1}$, the oblique wave propagates at $11\frac{1}{2}°$ to the wind direction and $\Delta\omega = 0{\cdot}12\,\mathrm{sec}^{-1}$. (The period $2\pi/\Delta\omega$ is then only about a minute, but it becomes increasingly large as the angle of propagation of the oblique wave decreases.)

The velocity component u_3 in the x direction may be obtained from equation (3.2a). On evaluating the non-linear terms by the results of linear theory, we find that

$$\frac{\partial u_3}{\partial t} = \overline{u}'w_3 + \frac{il^2A_1A_2^*}{2km^2}\,e^{i(ly+\Delta\omega t)}\,e^{-kz}\frac{\partial}{\partial z}\left(\frac{e^{-mz}}{z+\alpha}\right).$$

If $u_3 = 0$ at $t = 0$, the appropriate solution is

$$u_3 = \frac{il^2A_1A_2^*}{m^4}\,e^{ily}\left[\overline{u}'t\left(\frac{f}{\Delta\omega}-h\right)+\frac{i(e^{i\Delta\omega t}-1)}{\Delta\omega}\left\{\frac{u'f}{\Delta\omega}-\frac{m^2e^{-kz}}{2k}\frac{\partial}{\partial z}\left(\frac{e^{-mz}}{z+\alpha}\right)\right\}\right].$$

Also, for sufficiently small $\Delta\omega$, a good approximation except very near $z = 0$ and $t = 0$ is

$$u_3 = \frac{il^2\bar{u}' A_1 A_2^*}{m^4(\Delta\omega)^2} e^{ily} f(z)\{\Delta\omega t + i(e^{i\Delta\omega t} - 1)\}.$$

For small values of $\Delta\omega t$, it is seen that u_3 is proportional to t^2, as was the case in result (3.14).

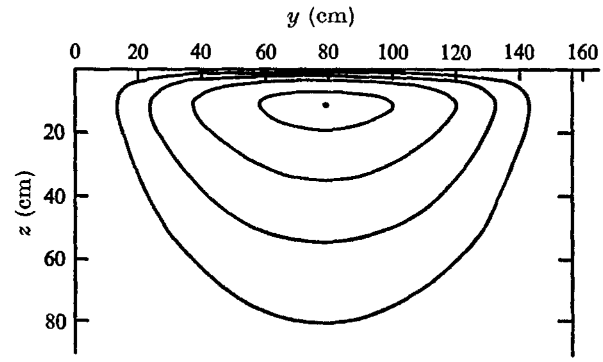

FIGURE 5. The lines $f(z)\cos ly$ = constant, for the non-symmetric interaction $m = 0{\cdot}1\ \mathrm{cm}^{-1}$, $l = 0{\cdot}02\ \mathrm{cm}^{-1}$, $\alpha = 20{\cdot}2$ cm.

5. The approximations

There are several approximations and assumptions implicit in the analyses of §§3 and 4. First, it was assumed that the properties of the interacting waves may be described by linearized theory. A necessary condition for this is that the wave-slopes are sufficiently small; that is,

$$\left(\frac{m}{\omega}\right)^2 |A_{1,2}|^2 \ll 1.$$

Secondly, it was assumed that the secondary flow produced by interaction of the two waves does not itself modify either of these waves by its interaction with the other. For example, there are non-linear terms proportional to $\exp[i(kx+ly-\omega t)]$ which involve products of the secondary flow (u_3, v_3, w_3) and the wave (u_2, v_2, w_2), but these have been neglected in evaluating the properties of the wave (u_1, v_1, w_1). Since the secondary flows found in §§3 and 4 grow with time, it is clear that this assumption must eventually be violated: in consequence, the analysis is valid only for sufficiently small times. More precisely, we require that

$$(t/\omega)\,|lA_{1,2}|^2, \quad t^2|lA_{1,2}|^2 \ll 1$$

in the analysis of §3, and that

$$\frac{|e^{i\Delta\omega t}-1|\,|lA_{1,2}|^2}{\omega\Delta\omega}, \quad \frac{\bar{u}'t|lA_{1,2}|^2}{\omega\Delta\omega} \ll 1$$

in the analysis of §4. (In deriving these conditions it was assumed that $\omega/\bar{u}'$ is $O(1)$ by virtue of the dispersion relationship (2.1).)

A third approximation was the adoption of an inviscid model, whereas, in practice, the influence of viscosity is significant in a layer near the surface. A further related approximation was the neglect of systematic tangential stress variations at the water surface which derive from the action of the airflow and of surface contamination. Thus, a viscous analysis would involve satisfying, at the water surface, an appropriate tangential-stress condition which takes these variations into account. For the secondary flows in question, the viscous boundary layer would penetrate to a depth $\delta \sim O[(\nu t)^{\frac{1}{2}}]$ beneath the water surface. Since, for the results of §§3 and 4 to provide reasonable approximations, it is necessary that δ is small compared with m^{-1}, we require that

$$m^{-1}(\nu t)^{\frac{1}{2}} \ll 1.$$

Subject to the above conditions, the analyses of §§3 and 4 constitute a self-consistent theory which describes the initial development of longitudinal vortex-type flows. These analyses seem capable of extension to velocity profiles which are other than linear, but considerable complications arise. In the interests of simplicity and brevity, such extensions are not attempted here. However, the same physical mechanism is sure to operate: the gravity waves distort the vorticity field associated with the primary flow, and their non-linear interaction causes a vorticity transfer from the primary flow to the secondary disturbance.

6. A steady viscous solution

For symmetric interactions of the type examined in §3, a steady-state solution may be found by incorporating the effect of viscosity. On including the viscous terms and assuming that the secondary flow is steady, the appropriate equations for the velocity components (u_3, v_3, w_3) are

$$\nu\left(\frac{\partial^2}{\partial z^2}-4l^2\right)u_3+\bar{u}'w_3 = [\mathbf{u}\,.\,\nabla u]_3, \tag{6.1}$$

$$-\nu\left(\frac{d^2}{dz^2}-4l^2\right)\left(\frac{d^2}{dz^2}-4l^2\right)\phi = 2il[\mathbf{u}\,.\,\nabla w]_3-\frac{\partial}{\partial z}[\mathbf{u}\,.\,\nabla v]_3, \tag{6.2}$$

$$v_3 = \frac{d\phi}{dz}e^{2ily}, \quad w_3 = -2il\phi\, e^{2ily},$$

where ϕ is now a function of z only. The right-hand sides of equations (6.1) and (6.2) are the same as those of (3.2a) and (3.3) respectively, and ν denotes the kinematic viscosity.

Since $\partial\zeta_3/\partial t$ is zero for steady flow and since the right-hand side of (3.6) is identically zero for the symmetric interaction considered, the kinematic boundary condition is

$$w_3 = 0 \quad (z = 0).$$

The normal-stress boundary condition equivalent to (3.5) now involves viscous terms; however, this boundary condition has no dynamic significance since it serves only to yield an expression for the surface displacement ζ_3. If the appropriate component of tangential stress is zero at the water surface—that is, if we may neglect such systematic stresses as may derive from the airflow and from surface contamination—we have the boundary condition

$$\frac{\partial w_3}{\partial y}+\frac{\partial v_3}{\partial z} = -\frac{1}{2}\left[\eta_2^* \frac{\partial}{\partial z}\left(\frac{\partial v_1}{\partial z}+\frac{\partial w_1}{\partial y}\right)+\eta_1 \frac{\partial}{\partial z}\left(\frac{\partial v_2^*}{\partial z}+\frac{\partial w_2^*}{\partial y}\right)\right] \quad (z=0).$$

It may be verified that, for symmetric interactions, the right-hand side of this equation is identically zero. Thus, the boundary conditions for ϕ are found to be

$$\phi = 0, \quad \frac{d^2\phi}{dz^2} = 0 \quad (z=0),$$

$$\phi, \phi' \sim 0 \quad (z \to \infty).$$

If, now, we define Φ by

$$-\frac{A_1 A_2^* il}{\nu m^4}\Phi = \left(\frac{d^2}{dz^2}-4l^2\right)\phi,$$

it is seen from (6.2) that Φ satisfies the equation and boundary conditions

$$\frac{d^2\Phi}{dz^2}-4l^2\Phi = G(z); \quad \Phi(0) = 0, \quad \Phi(\infty) = 0,$$

where $G(z)$ is as defined in (3.10) and the latter boundary condition results from the requirement that the vorticity perturbation tends to zero at large z. Clearly, the solution is

$$\Phi(z) = f(z),$$

where $f(z)$ is as given in (3.12). The function $\phi(z)$ therefore satisfies the equation and boundary conditions

$$\frac{d^2\phi}{dz^2}-4l^2\phi = -\frac{A_1 A_2^* il}{\nu m^4}f(z), \quad \phi(0) = 0, \quad \phi(\infty) = 0, \tag{6.3}$$

which has a solution identical in form to the right-hand side of (3.11), but with $G(z)$ replaced by the right-hand side of the present equation.

It is now a simple matter to state the explicit solution for $\phi(z)$. However, for brevity, we restrict attention to cases where $2(m-l)\alpha \gg 1$, when a much simpler approximate solution is valid. This approximate solution is found to be

$$\phi = -\frac{iA_1A_2^*(m^4+4k^2m^2-4k^4)}{8\nu m^4k^4}\left[\frac{l\,e^{-2mz}}{z+\alpha}-\frac{(l-k^2z)\,e^{-2lz}}{\alpha}\right], \tag{6.4}$$

which, when $iA_1A_2^*$ is taken to be real, yields streamlines of the form

$$\left[\frac{l\,e^{-2mz}}{z+\alpha}-\frac{(l-k^2z)\,e^{-2lz}}{\alpha}\right]\cos 2ly = \text{constant}.$$

Such streamlines are shown in figure 6 for the case $m = 0{\cdot}1\,\text{cm}^{-1}$, $l = 0{\cdot}02\,\text{cm}^{-1}$, $\alpha = 100\,\text{cm}$. Clearly, these are only qualitatively similar to the streamlines shown in figure 4 for the corresponding initial flow. As was to be expected, the diffusive role of viscosity has decreased the shear near the surface, and the maximum vertical velocities now occur at greater depth.

The velocity component u_3 is readily found from (6.1), where, as in §3, the right-hand side is identically zero. The appropriate equation and boundary conditions are then

$$\left(\frac{\partial^2}{\partial z^2}-4l^2\right)u_3 = -\frac{\bar{u}'w_3}{\nu},$$

$$u_3 = 0 \quad (z=\infty), \qquad \partial u_3/\partial z = 0 \quad (z=0),$$

where w_3 is known. If we again assume that $2(m-l)\alpha \gg 1$, the approximate solution is found to be

$$u_3 = \frac{\bar{u}'A_1A_2^*(m^4+4k^2m^2-4k^4)}{16\nu^2m^4k^4}\left[\frac{l^2e^{-2mz}}{k^2(z+\alpha)}+\frac{e^{-2lz}}{2\alpha}\left\{\left(1-\frac{k^2}{4l^2}\right)(1+2lz)-k^2z^2-\frac{2lm}{k^2}\right\}\right].$$

The above viscous analysis is likely to be valid only when the conditions

$$\left(\frac{m}{\omega}\right)^2|A_{1,2}|^2 \ll 1, \quad \frac{|A_{1,2}|^2}{\nu m^3\omega} \ll 1, \quad \frac{|A_{1,2}|^2}{(\nu m)^2} \ll 1$$

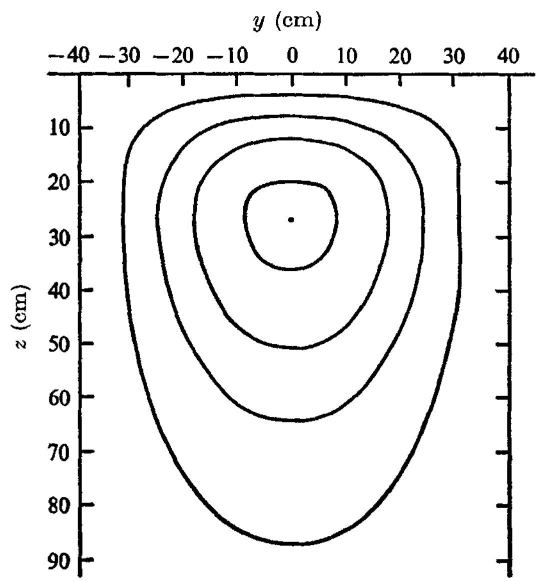

FIGURE 6. The approximate streamlines in the steady viscous case, for the symmetric interaction $m = 0{\cdot}1\ \mathrm{cm}^{-1}$, $l = 0{\cdot}02\ \mathrm{cm}^{-1}$, $\alpha = 100$ cm.

are satisfied. The last two conditions, which ensure that the secondary flow does not influence either of the primary waves through its interaction with the other, place a rather severe limitation on the amplitudes of the waves. One may imagine that, subject to these conditions, a smooth transition from the initial-growth solution of §3 to the present steady state would be accomplished by the gradual increase in importance of viscous diffusion.

7. The Stokes contribution

The foregoing analysis yields second-order Eulerian velocity components associated with particular wave interactions. However, the second-order Lagrangian velocity components are somewhat different owing to 'Stokes drift' contributions from the waves. These contributions are obtained by selecting those terms of appropriate periodicity from the expression for the Stokes velocity

$$\mathscr{U}_i = \overline{\left(\int_{t_0}^{t} u_j dt\right) \frac{\partial u_i}{\partial x_j}},$$

the bar denoting the average over a wave period and the velocity components $u_i \equiv (u, v, w)$ being given by linear theory. It is clear that $\mathscr{U}_i$ is independent of time, unlike the secondary flows of §§ 3 and 4, the former of which grows linearly in time. Consequently, after a sufficient time, the Eulerian velocities associated with these secondary flows are likely to be large compared with the Stokes velocities. Also, it is readily shown that, for the steady secondary flows examined in § 6, the Eulerian velocity components greatly outweigh the Stokes contribution when $\overline{u}'/k^2\nu \gg 1$.

A closer examination further reveals that symmetric interactions among waves of equal amplitude (i.e. those discussed in §§ 3 and 6 with $|A_1|$, $|A_2|$ equal) yield a Stokes contribution to the longitudinal vorticity which is identically zero.

The above facts indicate that the Stokes contribution to longitudinal vortex flows of the present type may usually be neglected. Such contributions may be calculated without undue difficulty if desired; but they are here omitted for the sake of brevity.

8. Discussion

Subject to the stated conditions, the analyses of §§ 3 and 4 describe the initial growth of longitudinal vortex flows owing to symmetric and non-symmetric wave interactions, while the analysis of § 6 depicts a steady vortex flow resulting from a symmetric wave interaction. Inevitably, however, these analyses stop far short of a complete description of the quasi-steady secondary motions which exist when wind blows over a water surface. The analyses concern only specific wave interactions; whereas, in practice, a whole spectrum of waves are present, many pairs of which may drive aperiodic or weakly periodic vortex motions. The final quasi-steady structure will be determined by a complicated non-linear interaction of *all* the components, and additional factors such as surface contamination, turbulence due to wave-breaking, thermal effects and the Coriolis force may all influence the flow.

This view is consistent with the suggestion of McLeish (1968) that windrows at sea may derive from the turbulence in the water. For a large part of the motion near the surface is attributable to a primary shear flow together with a continuous gravity-wave spectrum; and the present work shows how these ingredients may interact to produce the (unsteady) longitudinal vortices which are a characteristic feature of the turbulence.

A comparison of the secondary flows predicted by the present analysis and the existing observations of Langmuir vortices is necessarily rather tentative, in view of both the limitations of the analysis and the scarcity of reliable observational data. However, such a comparison is of some interest, for certain features of the observed phenomenon are fairly convincingly reproduced by the simple theoretical model.

Other features, of course, are not contained in the theoretical model. The asymmetries reported by Faller & Woodcock (1964) and Welander (1963), which may be attributable to the influence of the Coriolis force, are not represented. Nor does the analysis yield any prediction of a 'critical wind speed' at which windrows first occur. Such a critical speed has been reported by many authors, and it is undoubtedly true that rows are seldom observed at wind speeds below about $3\,\mathrm{m\,sec^{-1}}$. It seems likely that some secondary circulation due to wave interaction will occur at *all* wind speeds, provided there is a shear flow in the water. However, horizontal motion at the water surface is probably suppressed by surface contamination when the underlying circulation is weak. Windrows will first be observed when the circulations are sufficiently strong to compress the surface film into discontinuous bands.

The analyses of §§3 and 4 may serve to explain the rapid realignment of windrows following a sudden change in wind direction. For the necessary secondary flows will result as soon as the new primary shear flow is established in the upper layers of the water: there need not even be a substantial change in the wave spectrum, since suitable interactions may take place among the waves *already present*. These vortex motions will subsequently be reinforced as new waves are generated by the wind. (In fact, the analyses of §§3 and 4 may readily be extended to show that, for two primary waves growing as $e^{\sigma_1 t}$ and $e^{\sigma_2 t}$ respectively where σ_1 and σ_2 are small, the secondary flow grows approximately as $e^{(\sigma_1+\sigma_2)t}$.)

Presumably, the pattern of windrows will be most regular and parallel when suitable components of the wave spectrum are dominant. This may explain why windrows on lakes are usually more regular than at sea: the wave motion in lakes is likely to be less complex, and there may more readily emerge a single dominant vortex structure which determines the spacing of the windrows. However, the detailed prediction of this flow is outside the scope of this (or any) analysis. Indeed, it is impossible even to generalize the analysis of §3 to examine the initial development of the secondary flow resulting from interactions among a given wave spectrum: for, without a knowledge of the phase of each wave component, there is no way of telling whether contributions of particular interactions may reinforce or cancel each other. The nearest one can come to a prediction of row spacing is to discover, for a given gravity-wave spectrum, the fastest-growing component predicted by the analysis of §3 and also the dominant steady component predicted by the viscous analysis of §6. If the spanwise periodicities of these two components are in reasonable agreement, it is at least plausible that these may be closely related to the spacing of windrows.

Probably, the most appropriate criterion for determining the 'fastest-growing' component of §3 and the 'dominant' component of §6 is the longitudinal vorticity. Accordingly, we shall seek those disturbances for which §§3 and 6 predict the greatest growth-rate of vorticity and the greatest vorticity respectively.

First, we assume the presence of a continuous spectrum of wind-generated gravity waves. The associated linearized velocity components at each point

(x, y, z) may then be represented as Fourier integrals: in particular, we take the vertical velocity to be

$$w(x,y,z,t) = \int_{m=0}^{\infty}\int_{\theta=-\pi}^{\pi} A(m,\theta)\exp[-mz]\exp[i(kx+ly-\omega t)]\,dm\,d\theta,$$

where $m = (k^2+l^2)^{\frac{1}{2}}$, $\theta = \tan^{-1}(l/k)$ and $\omega(m,\theta)$ is given by the dispersion relation (2.1). Suppose, also, that

$$|A(m,\theta)|^2 = |A(m)|^2 G(\theta),$$

where $G(\theta)$ in an even function of θ whose value is unity at $\theta = 0$. Two such functions, each of which may provide an acceptable estimate for $G(\theta)$, are

$$G_1(\theta) = \cos^4(\tfrac{1}{2}\theta) \quad (0 \leqslant |\theta| \leqslant \pi), \quad G_2(\theta) = \cos^2\theta \quad (0 \leqslant |\theta| \leqslant \tfrac{1}{2}\pi)$$
$$= \quad 0 \quad (\tfrac{1}{2}\pi \leqslant |\theta| \leqslant \pi).$$

We now consider the initial growth of disturbances given by §3. [Note that the non-symmetric interactions of §4 yield secondary flows whose amplitudes are bounded and vary periodically in time, whereas the symmetric interactions of §3 drive secondary flows which grow in time until the eventual violation of the basic assumptions of the analysis. Accordingly, we can probably ignore the non-symmetric interactions when searching for the dominant vortex structure.]

It is readily seen from (3.3) that the growth-rate of longitudinal vorticity attains its greatest value at the free surface $z = 0$. When $(m-l)\alpha \gg 1$, this value is close to $B(m,\theta)\cos(2ly+\epsilon)$ where ϵ is some phase factor and

$$B(m,\theta) \equiv \frac{2kl(3k^2+l^2)\overline{u}'}{m^2\omega}|A(m)|^2 G(\theta).$$

Using the results $k = m\cos\theta$, $l = \sin\theta$ and setting $\omega \approx (gm)^{\frac{1}{2}}$, we obtain

$$B(m,\theta) = 2\overline{u}'g^{-\frac{1}{2}}m^{\frac{3}{2}}|A(m)|^2(1+2\cos^2\theta)\sin\theta\cos\theta\, G(\theta).$$

With $G(\theta)$ replaced by $G_1(\theta)$ defined above, $B(m,\theta)$ has a maximum with respect to θ at $\theta = 29°$; whereas, with $G(\theta)$ set equal to $G_2(\theta)$, the maximum occurs at $\theta = 26°$. These results are similar, and support the view that the optimum angle θ is not strongly dependent on the precise form of $G(\theta)$.

Now, if the wave energy is distributed according to the Neumann spectrum (see for example Kinsman 1965, chapter 8),

$$|A(m)|^2 = \omega^2|a(m)|^2 = \text{const}\times\omega^{-4}\exp[-2g^2\omega^{-2}U^{-2}] \quad (\omega \geqslant \omega_0),$$
$$= 0 \quad (\omega < \omega_0),$$

where U is the wind speed and ω_0 is a cut-off frequency which is dependent on fetch and time. Taking $\omega^2 = gm$, it is found that $B(m,\theta)$ has a maximum with respect to m at $m_1 = 2g/U^2$ provided $m_1 > m_0$, where $m_0 = \omega_0^2/g$ is the cut-off wave-number. However, in most cases of interest m_0 is likely to be large compared with m_1, and $B(m,\theta)$ then attains its greatest value close to the cut-off wave-number m_0.

We shall take the maximum value of $B(m,\theta)$ to occur close to $m = m_0$ and $\theta = 28°$. Then since $\theta = \sin^{-1}(l/m)$ and the spatial periodicity of the secondary disturbance is $\lambda = \pi/l$, the spacing associated with the fastest growing disturbance is

$$\lambda \approx \frac{\pi}{m_0 \sin\theta} = 6{\cdot}7 m_0^{-1}. \tag{8.1}$$

A similar procedure applied to the results of §6 yields the spacing of the steady disturbance with greatest vorticity. The vorticity, given by the right-hand side of equation (6.3), has magnitude $H(m,\theta,z) = \nu^{-1}|A(m,\theta)|^2 lm^{-4}f(z)$ where $f(z)$ is given approximately by (3.13). When $|m-l|\,\alpha \gg 1$, the greatest vorticity occurs at depth z_0 where $\exp[2(m-l)z_0] \approx m/l$. At this depth

$$H(m,\theta,z_0) = \frac{\overline{u}'|A(m)|^2 G(\theta)}{2\nu(gm)^{\frac{1}{2}}\cos\theta}(\sin\theta)^{1/(1-\sin\theta)}(1-\sin\theta)(1+\sin^2 2\theta),$$

approximately, which has a maximum with respect to θ at $\theta = 33°$ when $G(\theta) = G_1(\theta)$ and at $\theta = 28°$ when $G(\theta) = G_2(\theta)$. If, also, the maximum with respect to m again occurs close to the cut-off wave-number m_0, the preferred spacings are

$$\lambda \approx \frac{\pi}{m_0 \sin\theta} = \begin{Bmatrix} 5{\cdot}77\, m_0^{-1} & (\theta = 33°) \\ 6{\cdot}69\, m_0^{-1} & (\theta = 28°) \end{Bmatrix}. \tag{8.2}$$

Results (8.1) and (8.2) are in good agreement: we have therefore shown that the spacing associated with the fastest growing component is close to that of the greatest component occurring in the steady state. Despite the limitations of the analysis from which these results were derived, it is reasonable to expect that this spacing may correspond to that of observed windrows. The above results may be summarized by the statement that *the row spacing is approximately equal to the cut-off wavelength of the wave-energy spectrum.* This criterion seems to be consistent with existing observations.

However, the above criterion is not applicable when the spectrum is particularly well-developed, for the maxima with respect to m of $B(m,\theta)$ and $H(m,\theta,z_0)$ do not then occur close to the cut-off wave-number m_0. Instead, $B(m,\theta)$ attains its maximum at $m_1 = 2g/U^2$ where $m_1 > m_0$, and the associated spacing is $\lambda \approx 3{\cdot}3U^2/g$. The corresponding maximum of $H(m,\theta,z_0)$ may be shown to occur at $m_2 = 8g/7U^2$ provided $m_2 > m_0$, and the preferred spacing is then $\lambda \approx 6U^2/g$. At typical wind speeds, these spacings are 100 m or more, which estimates may furnish upper bounds for the spacing of windrows.

It may be concluded with some certainty that wave interactions of the type considered will usually play a substantial part in the formation of longitudinal vortices and, consequently, of windrows. However, it is desirable that further experiments be undertaken, comprising the simultaneous measurement of wind-speed, shear flow in the water, wave spectrum, and the spacing and structure of Langmuir vortices. In particular, such measurements would provide a test of the spacing criteria just derived.

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